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Spectrum and Eigenfunctions of the Frobenius–Perron Operator of the Tent Map

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The spectrum and eigenfunctions of the Frobenius–Perron operator induced by the tent map are discussed in detail. Special attention is paid to the case where the critical point of the map lies on an aperiodic trajectory and the differences from maps with a periodic critical trajectory are stressed. It is shown that the relevant eigenvalues of the spectrum are not sensitive to the aperiodicity of the critical trajectory. All other parts of the spectrum and all eigenfunctions in particular are changed drastically if the critical trajectory becomes aperiodic. The intimate connection between the point spectrum and the kneading invariant is established and the critical slowing down as well as the band splitting are a consequence of its properties. The structure of the infinite sequence of null spaces and its implications on the spectrum of the operator are discussed. It is shown that any initial distribution $P(0, x)$ of bounded variation can be projected uniquely onto the eigenfunctions of the relevant eigenvalues and that the time dependence of $P(n, x)$ is determined by this expansion up to an error $O(\eta^n)$. From this the stationary and the asymptotic behavior of the correlation function $\langle x(n)\, x \rangle$ can be derived exactly.

KEY WORDS: Chaos; tent map; Frobenius–Perron operator; null space; spectrum; left and right eigenfunctions; adjoint operator; invariant measure; critical slowing down; correlation functions; power spectrum.

1. INTRODUCTION

In the last half decade there has been a growing interest in nonlinear dynamical systems. The most outstanding examples of such systems are the Lorenz model[1] which has a continuous time parameter, and the logistic equation[2] with a discrete time parameter. A rich structure of subharmonic

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bifurcations, period doublings, universal functions, Cantor sets, noisy bands, and chaos has been found.[3] But in all these cases, a large part of the investigations had to be done numerically. Analytic results are often restricted to qualitative descriptions like the Sarkovskii ordering of periodic orbits of continuous maps with a single maximum or the existence of non-zero measures of maps everywhere expanding. So, I think, it is justified to look for examples where quantitative results can be derived analytically.

We will discuss the band splitting, the critical slowing down, and the long time tails of the correlation function $\langle x(n)\, x\rangle$ for the particularly simple, but nontrivial tent map:

$$x_{n+1} = f(x_n) = \begin{cases} ax_n, & x < 1/2 \\ a - ax_n, & x \geqslant 1/2 \end{cases} \tag{1.1}$$

in the entire parameter regime $a \in (1, 2]$. The behavior of this map has been discussed extensively by Derrida *et al.*[6] Our main interests are the spectrum and the eigenfunctions of the Frobenius–Perron operator H,

$$HP(x) = \sum_i |d/dx\, f_i^{-1}(x)|\; P(f_i^{-1}(x))\, \chi_{f_i^{-1}(x)} \tag{1.2}$$

where f_i^{-1} are the different branches of the inverse of $f(x)$ and $\chi_{f_i}^{-1}(x)$ is the indicator function of the branch i.

Having found the eigenfunctions $P_n(x)$ of H, we can calculate the correlation function $\langle x_n x\rangle$

$$\langle x(n)\, x\rangle = \int_I x(n)\, x d\mu = \int_I xH^n(xd\mu) \tag{1.3}$$

if there is an expansion

$$x\mu_0(x) = \sum_{i=0}^{\infty} \alpha_i P_i(x) \tag{1.4}$$

where $\mu_0(x) = P_0(x)$ is the invariant density and α_i are coefficients, still to be determined. We are specially interested, if this expansion makes sense for all parameter values of a and, if this is the case, how the coefficients α_i can be determined. (For a detailed description of a special parameter set see Mori *et al.*[4].)

The tent map has been investigated in great detail very recently by Mori and coworkers[5] for those parameter values a, where the trajectory of the critical point $x_c = 1/2$ (hereafter called the critical trajectory) is periodic or falls onto a periodic orbit in finite times. Then, the invariant density is piecewise constant and has only a finite number of steps. The steps define the boundaries of mutually disjoint intervals. The tent map is now equivalent to a finite Markov chain and the problem of finding spectrum and eigenfunctions of the Frobenius–Perron operator is reduced to linear algebra in a finite-dimensional vector space. However, the set A_M of parameter values a with this property is of measure zero. (We will call this the periodic case.) Although Mori *et al.*[5] found numerically, that the time correlation function and the power spectrum are insensitive to small deviations from the periodic case, it is certainly worthwhile to investigate whether the predictions made for maps with periodic critical trajectories carry over to maps with an aperiodic critical trajectory. The special case of the tent map with $a=2$ has been recently discussed by Grassberger[12] for the special case of analytic densities.

We will discuss in the following the spectrum of H with respect to various function spaces:

I. $L_2(\mu)$: Space of functions f defined on $(0, 1)=I$ such that

$$\int_I |f|^2 \mu(x)\, dx < \infty$$

where $\mu(x)$ is a positive semidefinite measure density.

II. $L_2(\mu)\,|_{BV}$: Space of functions from $L_2(\mu)$ such that $\mathrm{Var}(f) < \infty$.

III. $LP(\mu)$: Space of piecewise polynomial functions with jumps only at the points $\{f^n(0)\}$.

IV. $L\theta(\mu)$: Space of piecewise constant functions with jumps only at the points $\{f^n(0)\}$.

The number of jumps is finite if the critical trajectory is periodic and infinite otherwise. Eigenfunctions of H which are elements of $L\theta(\mu)$ will be called relevant (for reasons which will become clear later) and their eigenvalues λ_i are called relevant eigenvalues if $|\lambda_i| > 1/a$. The norm we use for elements of $L\theta(\mu)$ is

$$\|p(x)\|_\theta := \mathrm{Var}(p(x)) + p(0^+) \tag{1.5}$$

Restricting the domain of H to $L\theta(\mu)$, the first part of the results can be summarized in the following way:

(i) The relevant part of the spectrum of the Frobenius–Perron operator is not sensitive to whether $a \in (1, 2]$ is an element of A_M or not: If $a \in A_M$, one can always find a finite interval $[a, a+\delta_0]$ such that the shift of an eigenvalue $|\lambda_a - \lambda_a + \delta_0|$ is smaller than a given ε, provided the modulus of the eigenvalue λ_a is larger than $1/a$.

Furthermore, one can find another δ, usually very much smaller than δ_0, so that no additional eigenvalue will be found with modulus larger than

$1/(a-\varepsilon)$. We will discuss these statements for the case $a=\sqrt{2}$ (i.e., the first band-splitting transition). The generalization to other values of $a \in A_M$, is straightforward.

(ii) The low-lying eigenvalues, however, are changed dramatically. If $a \in A_M$, the spectrum of H consists of a pure point spectrum and the point $\lambda = 0$, which is infinitely degenerate. If $a \notin A_M$, the point spectrum with modulus smaller than $1/a$ vanishes and instead we get a continuous spectrum at $=1/a$.

(iii) Allowing $LP(\mu)$ as domain, the operator H has polynomial-like eigenfunctions of degree $2n$ with nonzero eigenvalues, if $a \in A_M$. If $a \notin A_M$, the only eigenfunctions with eigenvalues $\lambda \neq 0$ are stepwise constant functions.

(iv) If we consider the larger function space $L_2(\mu)$, every complex number $|\lambda| < 1$ is eigenvalue of H for an infinite number of eigenfunctions irrespective of whether the critical trajectory is periodic.

(v) If $a \in A_M$, H can be restricted to a compact operator H which acts on the finite-dimensional space $L\theta(\mu)$ and the adjoint operator H_c^+ has the same nonzero spectrum as H_c. This is not possible, if $a \notin A_M$. Then, H is not compact, the adjoint operator has no null space, and the only eigenvalues of H^+ are roots of 1.

(vi) For any $h(x) \in L_2(\mu)\,|_{BV}$ there is a unique projection of $h(x)$ onto the relevant eigenfunctions $p_n(x) \in L\theta(\mu)$ such that

$$\|H^n h(x) - \sum_{i=0}^{m} \alpha_i(\lambda_i)^n\, p_i(x)\| \leqslant c\eta^n \tag{1.6}$$

where $|\lambda_m| > \eta > 1/a$, λ_i the $m+1$ largest relevant eigenvalues and α_i are constants determined by a Riemann–Stieltjes integral over $h(x)$. The projection algorithm works although there are no left eigenfunctions for eigenvalues $|\lambda_i| \neq 1$.

(vii) The nondecaying part of the correlation functions can be calculated exactly for all parameter values a. Asymptotically, the power spectrum does not depend on the distinction $a \in A_M$ or $a \notin A_M$.

We should mention that the exact invariant density for all parameter values a has already been derived by Derrida *et al.*[6] as a consequence of their λ expansion. The calculations of this paper follow a path opposite to that of Derrida *et al.* They started from the λ expansion and found the invariant density. We, however, find from a general ansatz for *all* steplike eigenfunctions the λ expansion as the characteristic polynomial of the relevant point spectrum of the Frobenius–Perron operator.

The paper is organized as follows: In Section 2 we define the problem, briefly discuss the null space of the Frobenius–Perron operator, make a general step function ansatz for the eigenfunctions of H, and get the λ expansion as the characteristic polynomial. In Section 3, the characteristic polynomial is discussed and we show how the point spectrum depends on the control parameter a near the band-splitting point $a=\sqrt{2}$. Rigorous upper bounds for the shift of the eigenvalues are given and the critical slowing down at points $a=2^{2^{-n}}$ as well as the band splitting are discussed briefly. In Section 4, we show that in the aperiodic case no polynomial-like eigenfunctions exist and that the circle $|\lambda a| \leqslant 1$ belongs to the continuous spectrum. We give an expression for the approximate eigenfunctions and discuss finally the consequences of the infinite sequence of null spaces. In Section 5 we discuss the band splitting, the spectrum of the adjoint operator H^+, and the left eigenfunctions. In Section 6, we show how an initial distribution $h(x) \in L_2(\mu)\,|_{BV}$ can be expanded in terms of eigenfunctions from $L\theta(\mu)$ and prove Eq. (1.6). In Section 7 we derive an expression for the nondecaying part correlation function $\langle x(n)\, x\rangle$ as well as its power spectrum. Section 8 contains the conclusions.

2. THE FROBENIUS PERRON OPERATOR OF THE TENT MAP

The geometric structure of the tent map makes it obvious that the intervals $[0, f^2(x_c))$ and $(f(x_c), 1]$ are irrelevant for the statistical analysis, because the invariant density vanishes there. Thus it is convenient to perform a linear transformation of Eq. (1.1) mapping the interval $[f^2(x_c), f(x_c)]$ to $[0, 1]$. The transformed map reads

$$x_{n+1}=f(x_n)=\begin{cases} ax_n-a+2, & x_n<(a-1)/a \\ a-ax_n, & x_n \geqslant (a-1)/a \end{cases} \tag{2.1}$$

From now on, the symbol f will always refer to the transformed map of Eq. (2.1). The critical point lies at $x_c=(a-1)/a$, the first and second images of x_c define the boundaries of the interval $[f(x_c)=1$ and $f^2(x_c)=0]$. (See Fig. 1.) The control parameter a can take any value from the interval $(1, 2]$. Since the transformation between the tent map [Eq. (1.1)] and the transformed map [Eq. (2.1)] is one-to-one, all results can easily be transcribed to the original map.

The Frobenius–Perron operator induced by map (2.1) has the form

$$HP(x)=\frac{1}{a}P\left(1-\frac{x}{a}\right)+\frac{1}{a}P\left(\frac{x}{a}-\frac{2}{a}+1\right)\theta(x-2+a) \tag{2.2}$$

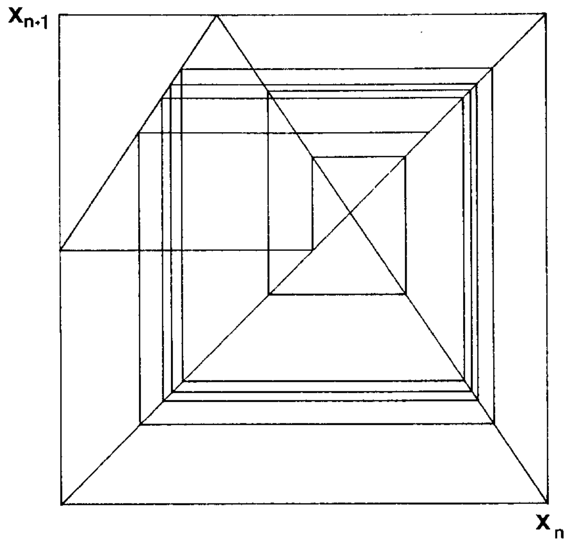

Fig. 1. The modified tent map is shown ($a = 1.48$) and the first few iterates of the critical trajectory are indicated.

where $\theta(x)$ is the stepfunction

$$\theta(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} \tag{2.3}$$

We are looking for solutions of the eigenvalue equation

$$\lambda P_\lambda(x) = HP_\lambda(x) \tag{2.4}$$

where $P_\lambda(x)$ denotes the eigenfunction P_λ with eigenvalue λ. As the domain of H, we allow $L_2(\mu)$.

As a first point, we note that H has a large null space: Applying H on the function $P_0(x)$

$$P_0(x) = P_{h_0}(x) = h_0(x - x_c)\, \theta(x - f_-^{-1} f(0)) \tag{2.5}$$

[$f_-^{-1}(x)$ and $f_+^{-1}(x)$ are the preimages of the branches of f with negative or positive slope, respectively] we get

$$HP_0(x) = 1/a\{h_0(1 - x/a) + h_0(x - 1/a)\}\, \theta(x - f(0)) \tag{2.6}$$

If $h_0(x)$ is an arbitrary antisymmetric function, then P_0 is an element of the null space N_0 of H. We can construct a sequence of null spaces of index k, with the property

$$N_k = HN_{k+1} \tag{2.7}$$

Thus, the null space of index k is mapped to the space N_0 after k iterations:

$$0 \xleftarrow{H} N_0 \xleftarrow{H} N_1 \xleftarrow{H} N_2 \xleftarrow{H} N_3 \dots \tag{2.8}$$

The most general element of N_1 is

$$P_{h_1}(x) = h_1(x - f_-^{-1}(x_c))\, \theta(x - f_-^{-2} f(0)) \tag{2.9}$$

where h_1 is antisymmetric. The null space N_2 is a sum of two independent elements, because now both branches of the inverse contribute:

$$P_{h_2}(x) = h_{21}(x - f_+^{-1} f_-^{-1}(x_c))\, \theta(x - f_+^{-1} f_-^{-2} f(0)) + h_{22}(x - f_-^{-2}(x_c))\, \theta(x - f_-^{-3} f(0)) \tag{2.10}$$

In order to find eigenfunctions of H with nonzero eigenvalue, we try an ansatz of the form

$$P_\lambda(x) = a_0 + \sum_{i=1}^{\infty} a_i \theta(S_{i-1}[x - f^i(0)]) \tag{2.11}$$

where $\{a_i\}$ and $\{S_i\}$ are constants to be determined. We note, that this is the general expression for an element from $L\theta(\mu)$. This ansatz is very natural: if we start with a continuous function and iterate it k times, we will end up with jumps at all points $x_i = f^i(0)$, $i \in \{0, \dots, k\}$. Starting, for example, with a constant function, we will get an expression like Eq. (2.11). If our initial test function was a polynomial of degree n [i.e., an element from $LP(\mu)$], we would find a piecewise polynomial function with jumps at the points x_i. These cases will be discussed in Section 4.

Applying the Frobenius–Perron operator H on $\theta(S_{i-1}[x - f^i(0)])$ we obtain

$$H\theta(S_{i-1}[x - f^i(0)]) = \frac{1}{a}\, \theta(S_i[x - f^{i+1}(0)]) + \frac{1}{2a}(1 - S_{i-1})\, \theta(x - f(0)) + \frac{1}{a} S_{i-1} \theta(x_c - f^i(0)) \tag{2.12}$$

where S_i is defined recursively

$$S_i = S_{i-1} \cdot \operatorname{sgn}(x_c - f^i(0)) \tag{2.13}$$

and $S_0 = 1$. Thus $\{\theta(S_{i-1}[x - f^i(0)])\}_\infty$ is an ordered sequence of θ functions which are mapped into each other sequentially. Each mapping produces additionally a constant and the first function in the sequence $\theta(x-f(0))$. From the mapping [Eq. (2.12)] it follows that the constants a_i of Eq. (2.11) have to fulfill the following relation:

$$a_{i+1} = z \cdot a_i, \qquad i > 2 \tag{2.14}$$

where $z = 1/(\lambda \cdot a)$. Equation (2.14) is solved easily: $a_i = z^{i-1} a_1$ $(i > 1)$. a_0 and a_1 are determined by two coupled homogeneous equations:

$$\begin{aligned} a_0 &= a_0 z + a_1 \sum_{i=1}^{\infty} z^i S_{i-1} \theta(x_c - f^i(0)) \\ a_1 &= a_0 z + a_1 \sum_{i=1}^{\infty} z^i \tfrac{1}{2}(1 - S_{i-1}) \end{aligned} \tag{2.15}$$

The zeros of the determinant of this system are eigenvalues of the operator H:

$$0 = \sum_{i=0}^{\infty} \sigma_i z^i \tag{2.16}$$

where $\sigma_i = S_{i-2}$, $\sigma_0 = -1$, and $\sigma_1 = 1$. The σ sequence is the kneading invariant of the critical point

$$\sigma_i = \sigma_{i-1} \operatorname{sgn}(x_c - f^i(0)) \tag{2.17}$$

where we define $\operatorname{sgn}(0) := -1$ and $\sigma_0 = -1$. The results are independent of the sign we attribute to the slope of $f(x)$ at $x = x_c$, but the choice -1 is the simplest one we can take. Setting $z = 1/a$, the series (2.16) is exactly the λ expansion of Derrida *et al.*[6] They derive from it the ordering of periodic cycles and give an expression for the invariant density. They showed also that the smallest real solution of Eq. (2.16) is $1/a$ for a given kneading sequence. In our derivation, this follows at once from the Frobenius–Perron theorem (and its generalization to function space operators[7,8]), which states that the spectrum of H lies within or perhaps on the unit circle of the complex λ plane and $\lambda = 1$ is always an eigenvalue. Here, this implies that $|z| = 1/a$ is a lower bound of all zeros of Eq. (2.16) and $z = 1/a$ is a solution, which is in complete agreement with the results of Derrida *et al.*[6] (See Figs. 2 and 3.)

Making use of Eqs. (2.15) and (2.16) we obtain the unnormalized eigenfunctions with eigenvalue λ

$$P_\lambda(x) = \frac{1}{2z} \frac{1-2z^2}{1-z} + \sum_{i=1}^{\infty} z^i \theta(S_{i-1}[x - f^i(0)]) \tag{2.18}$$

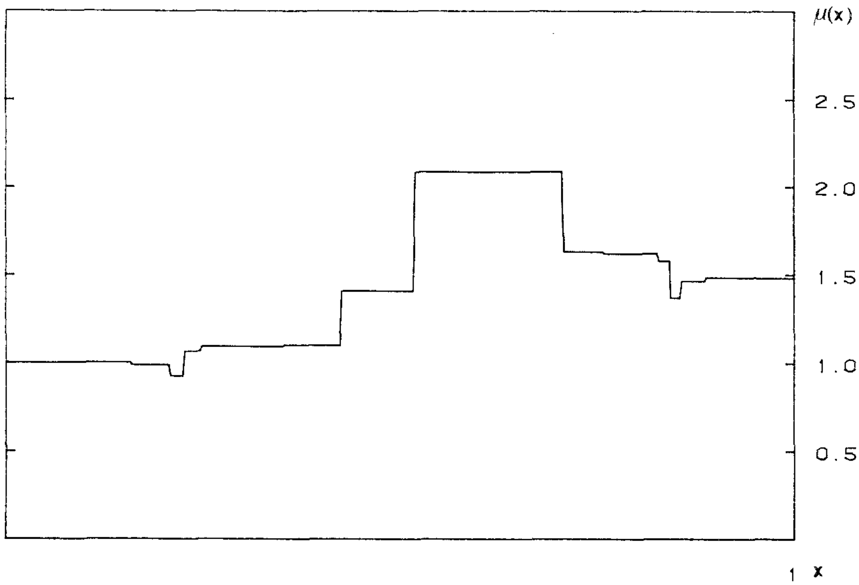

Fig. 2. The unnormalized stationary probability density μ is obtained from Eq. (2.11) for the parameter value $a = 1.48$.

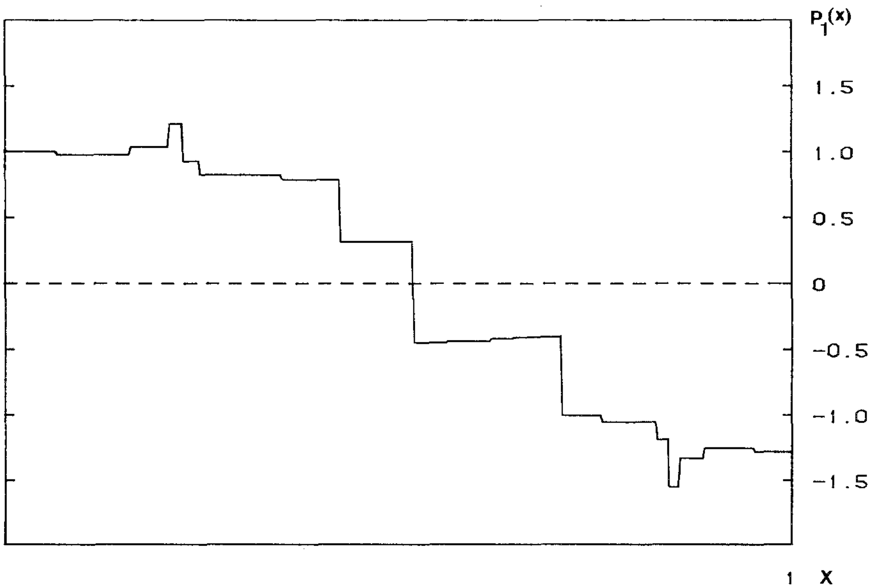

Fig. 3. The eigenfunction $P_1(x)$ with eigenvalue $\lambda_1 = -0.86723$ is shown. The control parameter is again $a = 1.48$.

where $z = 1/(\lambda a)$. The normalized invariant density reads

$$\mu(x) \equiv P_1(x) = \Gamma^{-1} \left\{ \frac{1}{2} \frac{a^2 - 2}{a - 1} + \sum_{i=1}^{\infty} a^{-1} \theta(S_i[x - f^i(0)]) \right\}$$

where $\Gamma = a - \sum_{i=1}^{\infty} a^{-i} S_{i-1} f^i(0)$

The invariant density is nonzero and positive definite on the whole interval, as long as $a > \sqrt{2}$. We will show in Section 5 that $\mu(x)$ is positive semidefinite for all $\alpha \in (1, 2]$. The variation of all eigenfunctions [Eq. (2.18)] is bounded.

3. THE RELEVANT POINT SPECTRUM OF THE FROBENIUS–PERRON OPERATOR

In this section we discuss the consequences of Eq. (2.16), which determines the spectrum of a Frobenius–Perron operator H whose domain is restricted to the function space $L\theta(\mu)$.

It is certainly not possible to find a general solution of the characteristic polynomial [Eq. (2.16)] of H for an arbitrary kneading sequence. On the other hand, if $\{\sigma_i\}$ is periodic or eventually periodic we can sum up the infinite series and find a finite number of eigenvalues. If, for example, 0 is mapped on the critical point, the itinerary of x_c has the form

$$I = \{\text{sign}(x_c - f^i(x_c))\} = - - + - - + - - + - - + - - \cdots \tag{3.1}$$

and the kneading invariant [compare Eq. (2.17)]

$$K = - + + - + + - + + - + + \cdots \tag{3.2}$$

so that the characteristic polynomial is reduced to

$$0 = \frac{-1 + z + z^2}{1 - z^3} \tag{3.3}$$

and we find $\lambda_1 = 1$, $a = (\sqrt{5} + 1)/2$ and $\lambda_2 = -(\sqrt{5} - 1)/(\sqrt{5} + 1)$. If, on the other hand, 0 is mapped to the fixed point, the itinerary of x_c must be

$$I = - - + - - - - \cdots \tag{3.4}$$

and the kneading invariant

$$K = - + + - + - + - + \cdots \tag{3.5}$$

The characteristic polynomial can be summed up, and we find

$$0 = \frac{-1 + 2z^2}{1 + z} \tag{3.6}$$

with the obvious solution

$$a = \sqrt{2}, \qquad \lambda_1 = 1, \qquad \lambda_2 = -1 \tag{3.7}$$

i.e., the tent map is not mixing any longer. The periodic case has been investigated in great detail[5] and we will use it only as a particular simple example of our more general analysis. The condition that x_c lies on a periodic orbit is only fulfilled for a subset A_M of the parameter interval (1, 2) which has measure zero. So it is an important question, how the spectrum and the eigenfunctions of H change, if we disturb the parameter value a leading to a periodic orbit of x_c by a small amount ε. We will pay particular attention to the point $a = \sqrt{2} \in A_M$, because at this point the invariant density becomes positive semidefinite and a second eigenvalue of H becomes slow. There is an infinite sequence of similar points at $a = 2^{2^{-M}}$, $M \in \mathbb{N}$, but the behavior of the map at these points can be reduced to the case $a = \sqrt{2}$.

Investigating the aperiodic neighborhood of $a = \sqrt{2}$, we will use Rouché's Theorem[9] several times: Let us suppose we have two complex functions $f_1(z)$ and $f_2(z)$, holomorphic on an open, simple connected region D and we can ensure that $|f_1(z)| > |f_2(z)|$ on the entire boundary C of D. Then $f_1(z)$ has the same numbers of zeros as $f_1(z) + f_2(z)$ within C.

The characteristic polynomial [Eq. (2.16)] is well defined for all z within the unit circle of the complex plane.

If the polynomial is finite (that means, the critical point belongs to a periodic orbit), one can show that all zeros of Eq. (2.16) are bounded above by $|z| = 2$: Using Rouché's theorem, one looks for a circle $|z| = \text{const}$ such that

$$\left| \sum_{i=0}^{N-1} \sigma_i z^i \right| < |z|^N \tag{3.8}$$

is guaranteed along the circle. Independently of N, we find

$$\left| \sum_{i=0}^{N-1} \sigma_i z^i \right| < \frac{|z|^N - 1}{|z| - 1} = \frac{|2+\gamma|^N - 1}{1+\gamma} < |2+\gamma|^N = |z|^N \tag{3.9}$$

if $|z| = 2 + \gamma$ and $\gamma > 0$. This defines a bound for all eigenvalues of stepwise constant eigenfunctions of H with x_c periodic:

$$1 > |\lambda| > 1/2a \tag{3.10}$$

Next, we want to investigate the spectrum near the periodic case $a=\sqrt{2}$. The itinerary of x_c at this point is

$$I_{a=\sqrt{2}} = --+----\cdots = --+(-)^{\infty} \tag{3.11}$$

The negative subsequences of I will be denumerated 1, 2, 3..., starting from the first $(+)$. So $I_{\sqrt{2}}$ has only a first infinite negative sequence. If we are near $a=\sqrt{2}$, the itinerary I_a will have a very long but finite first sequence of $(-)$, followed by $(+)$, then again a certain number of $(-)$ followed by $(+)$ and so on. If $a<\sqrt{2}$, it is clear from the geometry of the map that all negative sequences must have an odd number of elements followed by a single $(+)$, so that the most general expression for I with $a<\sqrt{2}$ is

$$I_{a<\sqrt{2}} = --+-(--)^{k_1}+-(--)^{k_2}+-(--)^{k_3}\cdots k_1>k_i \tag{3.12}$$

where the first negative sequence is the largest of all sequences. Then the kneading invariant reads

$$K_{a<\sqrt{2}} = -+(+-)^{k_1+1}(-+)^{k_2+1}(+-)^{k_3+1}\cdots \tag{3.13}$$

and the characteristic polynomial

$$\rho_a(z) = \sum_{i=0}^{\infty} \sigma_i z^i$$

can be written

$$\rho_a(z) = (1-z)\,\rho_{a^2}(z^2) \tag{3.14}$$

where the kneading sequence belonging to ρ_{a^2} is

$$K_{a^2<2} = -(+)^{k_1+1}(-)^{k_2+1}(+)^{k_3+1}\cdots \tag{3.15}$$

Looking for the zeros of $\rho a(z)$, we get

$$z=1 \qquad \text{and} \qquad \rho_{a^2}(z^2)=0 \tag{3.16}$$

where $\rho_{a_2}(z)$ is the characteristic polynomial of a map $f_{a'}$ with control parameter $a'=a^2>\sqrt{2}$. Thus the spectrum of H_a is obtained taking the roots of all eigenvalues of H_{a^2} and adding the additional point $\lambda=1/a$ to the spectrum. Because there are now two eigenvalues with modulus 1 ($\lambda_0=1$ and $\lambda_1=-1$), the flow is no longer mixing.

This procedure can be repeated at each point $a=2^{2^{-N}}$ and we get the characteristic polynomial

$$\rho(z) = \rho(z^{2^N}) \prod_{\kappa=0}^{N-1} (1-z^{2^\kappa}), \qquad 2^{2^{-N-1}}<a<2^{2^{-N}} \tag{3.17}$$

As far as the relevant spectrum is concerned, we can restrict ourselves to the parameter interval $a \in [\sqrt{2}, 2]$, because the following intervals $a \in [-2^{2^{-N+1}}, 2^{2^{-N}}]$ can be rescaled to $[2, \sqrt{2}]$. The most striking object in the neighborhood of $a = \sqrt{2}$ is the appearance of a second slow eigenvalue $\lambda = -1$.

If $a > \sqrt{2}$, the geometry of the map ensures that the most general itinerary has the form

$$I_{a>\sqrt{2}} = --+(--)^N(+)(-)^{M_1}(+)(-)^{M_2}, \quad \frac{\sqrt{5}+1}{2} > a > \sqrt{2} \tag{3.18}$$

where N is a nonzero number which goes to infinity as a approaches $\sqrt{2}$. As long as a is smaller than the golden mean, a symbol $(+)$ is always enclosed by two symbols $(-)$. The kneading invariant takes the form

$$K_{a>\sqrt{2}} = -++(-+)^N + (-+)^{l_1} \begin{Bmatrix} --(+-)^{l_2} \\ \text{or} \\ +(-+)^{l_2} \end{Bmatrix} + \cdots \tag{3.19}$$

where l_i is the integer part of $M_i/2$. We subtract formally the kneading sequence $K_{a=\sqrt{2}}$ and get

$$\tfrac{1}{2}[K_{a>\sqrt{2}} - K_{a=\sqrt{2}}] = (0)^{2N+3} + (-+)^{l_1}, \qquad \begin{Bmatrix} -0(00)^{l_2} \\ \text{or} \\ 0(00)^{l_2} \end{Bmatrix} \cdots \tag{3.20}$$

Keeping only the leading order, the characteristic polynomial reads

$$\rho_{a>\sqrt{2}}(z) = \frac{-1+2z^2}{1+z} + 2z^{2N+3}\frac{1-z^{2(l_1+1)}}{1+z} + \cdots \tag{3.21}$$

We are now looking for the zeros of the infinite series

$$(1+z)\,\rho_a(z) = -1 + 2z^2 + 2z^{1+n} - 2z^{3+n+2l_1} + \cdots \tag{3.22}$$

where $n = 2N+2$ tells us how many iterations it takes until $f^n(x_c)$ crosses to the left-hand side of x_c for the second time.

Making use of Rouché's theorem, the influence of the aperiodicity of $f^n(x_c)$ on the zeros of Eq. (3.22), can be estimated. We have to look for closed curves, which satisfy the inequality

$$\frac{2\,|z|^{n+1}}{1-|z|^2} < |-1+2z^2| \tag{3.23}$$

(We used the fact that all l_i are larger than or equal to 1.) First, we choose a small circle around $z = \pm 1/\sqrt{2}$ with radius ξ. Then we have to find an n, such that Eq. (3.23) is guaranteed. After some algebraic manipulations, we obtain

$$n+1 > \frac{\ln 2/|\xi| - \ln(2-\sqrt{2})}{\ln\sqrt{2} - \ln(1+\sqrt{2}\,\xi)}, \qquad \xi < 0.15 \tag{3.24}$$

Inserting $\xi = 1/10$, $1/100$, $1/1000$, we find that the trajectory of the critical point, periodic or aperiodic, must not cross to the left-hand side of x_c before $n = 13$, 15, and 21 iterations, respectively, in order to ensure that the distance between the smallest zeros of $\rho(z)$ and $\pm 1/\sqrt{2}$ is smaller than ξ. n grows logarithmically with $1/|\xi|$, which implies that the relevant eigenvalues of H are insensitive, to whether x_c lies on a periodic orbit.

The situation is different, if we look for an $\varepsilon > 0$, such that no additional zeros of $\rho(z)$ appear within a circle with radius $|z| = 1 - \varepsilon$ of the complex plane. Starting again from Eq. (3.23), we find that n has to satisfy the inequality

$$n(\varepsilon) > 7 + \frac{\ln(2/\varepsilon)}{\varepsilon} \tag{3.25}$$

in order to ensure inequality (3.23) along the circle $|z| = 1 - \varepsilon$. The outer circle is much more sensitive to a small deviation from the value $a = \sqrt{2}$ than the two eigenvalues ± 1. (i.e.,$z = \mp 1/\sqrt{2}$). To make sure that there are no other eigenvalues with modulus larger than $2\sqrt{2}/3$ ($\varepsilon = 1/4$), the trajectory of the critical point must not cross to the left-hand side of x_c before 15 iterations. Thus, given ε and ξ, there is always a small but finite parameter interval $a \in [\sqrt{2} + \delta, \sqrt{2}]$, such that the large eigenvalues of the Frobenius–Perron operator do not change considerably. $\delta(n)$ is approximately $\delta \approx ((\sqrt{2} - 1)/\sqrt{2})^{1/n}$.

There is a dense set of parameter values where x_c lies on a periodic orbit and the eigenvalues are known.[5] Shifting the control parameter a tiny amount from these values, one always finds that the relevant eigenvalues are not sensitive to the change in a. For the parameter values are dense, the intervals $[a + \delta, a - \delta]$ established by our inequalities can be made to overlap, and it is possible to follow the eigenvalues of H for larger deviation of a from $\sqrt{2}$.

Knowing that the relevant eigenvalues are continuous functions of a, we can investigate the critical slowing down of the eigenvalue $\lambda = -1$ near the band splitting, applying Newton's method to $\rho(z)$:

$$0 = -1 + 2z^2 + z^{1+n} - z^{1+n+l} + \cdots \tag{3.26}$$

Introducing $z = \mp 1/\sqrt{2} + \xi$, we find

$$\xi = -(2)^{-(n+4)/2} + (2)^{-(n+l+4)/2} \mp \frac{n+1}{2}(2)^{(2n+5)/2} \tag{3.27}$$

To leading order, both eigenvalues are shifted by the same amount. The critical slowing down is obtained solving the equations

$$\begin{aligned} \frac{1}{a} &= \frac{1}{a_c} - \xi \\ \frac{1}{a\lambda} &= -\frac{1}{a_c}\xi, \qquad a_c = \sqrt{2} \end{aligned} \tag{3.28}$$

for λ:

$$\begin{aligned} \lambda &= -1 + \frac{2}{a_c}(a - a_c) \qquad a > a_c \\ \lambda &= -1 \qquad\qquad\qquad\quad a < a_c \end{aligned} \tag{3.29}$$

The critical slowing down at higher-order band-splitting points $a_N = (2)^{2^{-N}}$ follows directly (compare 3.17) from Eq. (3.28) by rescaling the parameter a and taking the 2^N root:

$$\lambda = (-1)^{2^{-N}}(1 - \sqrt{2}\,[a^{2^{+N}} - a_c^{2^{+N}}]) \tag{3.30}$$

Equations (3.29) and (3.30) have already been derived in a different way in Ref. 5.

4. POLYNOMIAL EIGENFUNCTIONS, THE CONTINUOUS SPECTRUM AND THE NULL SPACES

So far, we have only investigated the very restricted class of stepwise constant eigenfunctions. The most general expression for these functions [Eq. (2.11)] is valid for periodic and aperiodic $f^n(x_c)$. If $f^n(x_c)$ is periodic, Eq. (2.11) can be summed up and, if $1/|\lambda a| > 1$, continued analytically to eigenvalues outside the unit circle. The system of eigenfunctions obtained in this way is certainly not complete. A natural extension of the very restricted function space $L\theta(\mu)$ would be to allow for piecewise polynomial functions from $LP(\mu)$ and to look for eigenfunctions of H in this larger space. If, for example, $a = \sqrt{2}$, we find the following eigenfunctions of H (there is actually an infinite number of piecewise polynomial eigenfunctions; we list only the first few of them):

1. Piecewise constant eigenfunctions:

$$\mu(x) = P_0^+ = \chi_1 + \sqrt{2}\,\chi_2, \qquad \lambda = 1 \tag{4.1a}$$

the invariant density, and

$$P_0^- = \chi_1 - \sqrt{2}\,\chi_2, \qquad \lambda = -1 \tag{4.1b}$$

2. Piecewise linear eigenfunctions

$$\begin{aligned} P_1 &= [2x-(2-\sqrt{2})]\,\chi_1, & \lambda &= 0 \\ P_{11} &= -2[2x-(3-\sqrt{2})]\,\chi_2, & P_1 &= HP_{11} \end{aligned} \tag{4.1c}$$

(these functions belong to the null space of H) and the piecewise quadratic eigenfunctions:

3. $$P_2^+ = \{x^2 - 2/3(3-2\sqrt{2})\}\,\chi_1 + \{(x-1)^2 - 1/3(3-2\sqrt{2})\}\,\sqrt{2}\,\chi_2, \qquad \lambda = 1/2$$

$$P_2^- = \{x^2 - 2/3(3-2\sqrt{2})\}\,\chi_1\{(x-1)^2 - 1/3(3-2\sqrt{2})\}\,\sqrt{2}\,\chi_2, \qquad \lambda = -1/2 \tag{4.1d}$$

where χ_1 and χ_2 are the indicator functions of intervals $(0, 2-\sqrt{2})$ and $(2-\sqrt{2}, 1)$, respectively. This system is sufficient to expand $x\mu(x)$ and $x^2\mu(x)$ in terms of eigenfunctions of H and to determine the correlation functions $\langle h(x_n)\,x\rangle$ and $\langle g(x_n)\,x^2\rangle$. The calculation of these and all other polynomial eigenfunctions can always be reduced to the problem of finding the eigenvalues and eigenvectors of finite-dimensional matrices or of solving an inhomogeneous system of linear algebraic equations. The set of eigenvectors we get in this way is obviously complete, and we can expand any function from $L_2(\mu)$ in terms of these eigenfunctions.

So, one might try to find corresponding sets in the case of aperiodic critical trajectories. A possible ansatz would be

$$P^{(N)}(x) = c_N + \sum_{i=1}^{\infty} \sum_{n=0}^{N} \{(s_i[x - f^i(0)])^{N-n}\,\alpha_n^i\}\,\theta(s_{i-1}[x - f^i(0)]) \tag{4.2}$$

The eigenvalues are determined by the highest power of x in the equation

$$\lambda P^{(N)}(x) = \frac{1}{a}\,P^{(N)}\left(1-\frac{x}{a}\right) + \frac{1}{a}\,P^{(N)}\left(\frac{x}{a} - \frac{2}{a} + 1\right)\theta(x-2+a) \tag{4.3}$$

In the linear case, $N=1$, the characteristic polynomial $\rho_a(z)$ can be summed up, and we get

$$\rho_a(z)=\frac{1}{1-z/a} \tag{4.41}$$

so that the eigenvalue equation reads

$$0=\frac{\lambda a^2}{\lambda a^2-1} \tag{4.4b}$$

which has only the trivial solution $\lambda=0$. Thus the linear polynomial and all other polynomials of odd degree belong to the null space.

As far as the even polynomials are concerned, we will restrict ourselves to the quadratic case:

$$P(x)=\sum_{i=0}^{\infty}\{\alpha_i[S_{i-1}(x-f^i(0))]^2+\beta_i[S_{ivr}\langle x-f^i(0))]+\gamma_i\}$$
$$\times\,\theta(S_{i-1}[x-f^i(0)]) \tag{4.5}$$

Inserting Eq. (4.5) into the eigenvalue Eq. (4.3) and comparing equal powers of x, we get the recursive relations

$$\alpha_i=w\alpha_i, \qquad \beta_{i+1}=aw\beta_i, \qquad \gamma_{i+1}=a^2w\gamma_i, \qquad i>1 \quad (4.6)$$

$w=1/(\lambda a^3)$, with the solutions

$$\alpha_i=w^{i-1}\alpha_1, \qquad \beta_i=(aw)^{i-1}\beta_1, \qquad \gamma_i=(a^2w)^{i-1}\gamma_1, \qquad i>1 \quad (4.7)$$

The eigenvalues λ are determined from the homogenous system of algebraic equations for α_0 and α_1. We get exactly the characteristic polynomial of the stepwise constant eigenfunctions [Eq. (2.16)] if we replace z by w. Thus, for every eigenvalue $\lambda^{(0)}$ of the stepwise constant system there is an eigenvalue $\lambda^{(2)}$ of stepwise quadratic eigenfunctions and they are related by

$$\lambda^{(2)}=\frac{\lambda^{(0)}}{a^2} \tag{4.8}$$

This statement extends to all polynomial eigenfunctions of degree $(2n)$

$$\lambda^{(2n)}=\frac{\lambda^{(0)}}{a^{2^n}} \tag{4.9}$$

Inserting Eq. (4.8) into Eq. (4.7), the coefficients of the expansion [Eq. (4.5)] read

$$\alpha_i=[1/(\lambda^{(0)}a)]^{i-1}\,\alpha_1, \qquad \beta_i=(1/\lambda^{(0)})^{i-1}\,\beta_1, \qquad \gamma_i=(a/\lambda^{(0)})^{i-1}\,\gamma_1 \quad (4.10)$$

We know from the Frobenius–Perron theorem,[2] that $|\lambda^{(0)}| < 1$ when $a \in (1, 2]$. Thus the partial sums of β_i and γ_i diverge even for $\lambda^{(0)} = 1$, unless the critical trajectory is periodic and the sums can be continued analytically: If x_c lies on an aperiodic orbit, there are no polynomial-like eigenfunctions of the form of Eq. (4.5). This is rather plausible, if we remember that the jumps of the ansatz (4.5) are dense on the interval (0, 1). So, we have no possibility of distinguishing between a polynomial eigenfunction of degree n and the stepwise constant eigenfunction by investigating the neighborhood of a given point. It makes sense, however, if there is only a finite number of jumps as in the case of a periodic critical trajectory.

So far, we have found only a finite number of eigenfunctions with eigenvalues $1 > |\lambda| > 1/a$, which is certainly not sufficient to expand an arbitrary initial probability distribution $P(x, 0) \in L_2(\mu)$. On the other hand, ansatz (2.11) is the most natural one, because it already mimics the forward iteration we want to apply on $P(x, 0)$ afterwards. The only way out is that there is an accumulation of eigenvalues at the circle $|z| = 1$ or that the circle belongs to the continuous spectrum of H. Near $|z| = 1$, the ansatz (2.11) is ill behaved because the infinite series converge badly. At $|z| = 1$, we can only hope to find a sequence of approximate eigenfunctions.

In this sense, we start with a well-defined function $P_N(x) \in L\theta(\mu)$ and see if it is reproduced under the application of H to some accuracy:

$$\begin{aligned} P_N(x) = {} & \frac{1}{\lambda a - 1} \sum_{i=1}^{N} \left(\frac{1}{\lambda a}\right)^i \theta(x_c - f^i(0))\, S_{i-1} \\ & + \sum_{i=1}^{N} \left(\frac{1}{\lambda a}\right)^i \theta(S_{i-1}[x - f^i(0)]) \end{aligned} \tag{4.11}$$

where λ is an arbitrary complex number at the moment. Applying H to it, we find

$$\begin{aligned} & aHP_N(x) - \lambda a P_N(x) \\ & \quad = \left(\frac{1}{\lambda a}\right)^N \theta(S_N[x - f^{N+1}(0)]) \\ & \qquad + \left\{\lambda a \sum_{i=0}^{N+2} \sigma_i^{z^i} - \left(\frac{1}{\lambda a}\right)^N \frac{1}{2}(1 - S_N)\right\} \theta(x - f(0)) \end{aligned} \tag{4.12}$$

Using the θ norm [compare Eq. (1.5)] and normalizing P_N(i.e., $\hat{P}_N = P/\|P\|_\theta$) the norm of Eq. (4.12) becomes

$$\|H\hat{P}_N(x)-\lambda\hat{P}_N(x)\|_\theta = \frac{\left|\frac{1}{\lambda a}\right|^N + \left|\lambda a \sum_{i=0}^{N+2} \sigma_i z^i - \left(\frac{1}{\lambda a}\right)^N \frac{1}{2}(1-S_N)\right|}{\frac{a}{|\lambda a - 1|}\left|\sum_{i=1}^{N}\left(\frac{1}{\lambda a}\right)^i \theta(x_c - f^i(0))\, S_{i-1}\right| + a\sum_{i=1}^{N}\left|\left(\frac{1}{\lambda a}\right)^i\right|} \tag{4.13}$$

If $|1/\lambda a| < 1$, the denominator of Eq. (4.13) stays finite in the limit of large N and we find

$$\lim_{N\to\infty} \|H\hat{P}_N(x)-\lambda\hat{P}_N(x)\|_\theta < \left(\frac{1}{|z|}-1\right)\left|\sum_{i=0}^{\infty}\sigma_i z^i\right| \tag{4.14}$$

From this, we obtain again the characteristic polynomial as a necessary condition for the point spectrum of H

$$\sum_{i=0}^{\infty}\sigma_i z^i = 0 \tag{4.15}$$

If we take $1/\lambda a = e^{i\rho}$, $\rho \neq 0$, Eq. (4.13) becomes

$$\|H\hat{P}_N(x)-\lambda\hat{P}_N(x)\|_\theta < \frac{1}{a}\,\frac{2+\left|\sum_{i=0}^{N+2}\sigma_i z_i\right|}{N-1} \tag{4.16}$$

and for very large N, we get

$$\|H\hat{P}_N(x)-\lambda\hat{P}_N(x)\|_\theta \approx \frac{1}{a}\frac{1}{N}\left|\sum_{k=0}^{N+2}\sigma_k e^{2\pi i k\rho}\right| \tag{4.17}$$

We call an aperiodic trajectory typical, if the ratio p of the numbers of $\sigma_k = 1$ and the numbers of $\sigma_k = -1$ is the same for all infinite subsequences $\{\sigma_{k\cdot\alpha_1+\alpha_2}\}_k$, $\alpha_1 > \alpha_2$ fixed, of the kneading sequence $\{\sigma_k\}$.

For such trajectories, Eq. (4.17) simplifies to

$$\|H\hat{P}(x)-\lambda\hat{P}(x)\|_\theta\,|_{\lambda=e^{i\rho}/a} \simeq \frac{1}{a}\left[\frac{p(1-p)}{N}\right]^{1/2} \xrightarrow[N\to\infty]{} 0 \tag{4.18}$$

Thus, the circle $|\lambda| = 1/a$ in the complex λ plane belongs to the continuous spectrum of H^{10} and

$$P_{N,\alpha}(x) = \frac{1}{e^{i\alpha}-1}\sum_{k=1}^{N} e^{-i\alpha k}\theta(x-f^k(0))\, S_{k-1} + \sum_{k=1}^{N} e^{-i\alpha k}\theta(S_{k-1}[x-f^k(0)]) \tag{4.19}$$

is an approximate eigenfunction of H. The right-hand side in Eq. (4.17) gives an estimate of the accuracy of the approximate eigenfunctions. They work particularly well if λ is periodic.

The case $|1/\lambda a| > 1$ only makes sense if we can ensure that

$$\sum_{i=0}^{N+2} \sigma_i z^i = 0 \tag{4.20}$$

But the addition of a new power of z will completely rearrange the zeros $z_i^{(N)}$, $|z_i^{(N)}| > 1$ and we cannot hope that $z_i^{(N)}$ will become stationary eventually, unless the critical trajectory is periodic.

Up to now, we have concentrated on functions with jumps at points $\{f^n(0)\}$. We have found that there is at least a finite number of eigenfunctions from $L\theta(\mu)$ and, under certain conditions, an infinite set of approximate eigenfunctions, also from $L\theta(\mu)$. All eigenvalues of these eigenfunctions lie in the interval $1 \geqslant |\lambda_i| \geqslant 1/a$. If the critical trajectory is periodic we find additionally stepwise polynomial eigenfunctions with eigenvalues $|\lambda| < 1/a$, which are not stable under small perturbations.

If we now allow for $L_2(\mu)\,|_{BV} \supset L\theta(\mu)$ as the domain of H we can try to expand an arbitrary element $h \in L_2(\mu)\,|_{BV}$ in terms of eigenfunctions from $L\theta(\mu)$ by projecting it into $L\theta(\mu)$. This should give us the asymptotic behavior of $H^n h(x)$.

But there are severe complications: If we take $L_2(\mu)$ [or $L_2(\mu)\,|_{BV}$] as domain of H, H is not compact irrespective of whether the critical trajectory is periodic. This has dramatic consequences for the spectrum of H.

For the sake of convenience, let us concentrate on the case of $a = 2$, where the critical trajectory ends in a fixed point. Conclusions made here can be transferred[11] to all band-splitting points $a = 2^{2^{-N}}$. In that case, the Frobenius–Perron operator reads

$$HP(x) = 1/2[P(1 - x/2) + P(x/2)] \tag{4.21}$$

$\mu(x) \equiv 1$ is the invariant density and Bernoulli's polynomials $B_{2n}(x/2)$ are eigenfunctions with eigenvalues 2^{-2n} (these are the polynomial eigenfunctions),

$$N_0 = \{\sin 2\pi n x \mid n \in \mathbb{N}\} \tag{4.22a}$$

is the null space of H, i.e., $HN_0 = 0$.

$$N_v = \{\cos 2\pi 2^{v-1} q x \mid v \in \mathbb{N}^+,\ q \in (\text{odd integers})\} \tag{4.22b}$$

are successive null spaces of higher index v with the property

$$N_v = HN_{v+1} \tag{4.22c}$$

We know from Fourier analysis, that the direct sum of all null spaces and the invariant density form a complete linear space. The infinite sequence of null spaces allows us to construct a uncountable number of eigenfunctions of H. The simplest family of these eigenfunctions reads

$$P_{\lambda,q}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi \cdot 2^{n-1} qx) \tag{4.23}$$

where q is an arbitrary odd number and λ any complex number with modulus smaller than 1. Inserting this expression in Eq. (4.21), we get

$$HP_{\lambda,q}(x) = \lambda P_{\lambda,q}(x) \tag{4.24}$$

because $\cos \pi qx$ can be expanded in terms of the functions of N_0 and, therefore, is mapped to zero. So, we find that every point in the interior of the disk $|\lambda| < 1$ belongs to the point spectrum H and is also an accumulation point. This is also true for all other a, for the argument depends only on the existence of an infinite sequence of null spaces. Usually, the eigenfunctions constructed in that way are not differentiable but still measurable. They differ in one peculiar point from Eq. (2.11). Whereas the eigenfunctions of Eq. (2.1) had their jumps in the forward direction, the null space based eigenfunctions have jumps or other discontinuities at the preimages of the critical point (i.e., in the backward direction).

It can be shown that eigenfunctions of this kind with eigenvalues of modulus $|\lambda| > 1/a$ have an unbounded variation.[11] (Actually, these functions have a capacity dimension $d_c > 1$.) Thus the restriction of H to $L_2(\mu)\mid_{BV}$ has a continuous spectrum

$$\sigma_c = \{z \in \mathbb{C} \mid |z| \leqslant 1/a\}$$

and a point spectrum $(1 \leqslant |\lambda| < 1/a)$. We also see that functions of unbounded variation can decay arbitrarily slowly to their stationary value. On the other hand, initial distributions of bounded variation will decay exponentially to the subspace $L\theta(\mu)$ of $L_2(\mu)\mid_{BV}$ because there is a finite distance between the continuous and the discrete part of the spectrum of H restricted to $L_2(\mu)\mid_{BV}$. In Section 6, we will show that this plausibility argument is correct.

5. BAND SPLITTING AND LEFT EIGENFUNCTIONS

In Section 2, the eigenfunctions $P_{a,\lambda}(x)$ have been derived, where λ is a discrete eigenvalue. Below the band-splitting point (i.e., $a > \sqrt{2}$) the invariant density $P_{a,1}(x)$ is nonzero on the whole interval and the jumps are dense on (0, 1). This is no longer the case if $a<\sqrt{2}$. The geometrical structure of the tent map reveals that there is an ordering of the itinerary of the critical trajectory as soon as $a<\sqrt{2}$:

$$f^{2n}(0)<f^2(0)<f(0)<f^{2n+1}(0) \tag{5.1}$$

for all positive $n>2$. This enables us to evaluate the eigenfunctions $P_{a,\lambda}(x)$ [see Eq. (2.18)] further: Inequality (5.1) ensures that there are no jumps of the eigenfunctions within the interval $[f^2(0), f(0)]$ and we get from Eq. (2.18)

$$P_{a,\lambda}(x)\,|_{[f^2(0),f(0)]}=\frac{1}{2}\frac{1}{z}\frac{1-2z^2}{1-z}+\sum_{i=1}^{\infty} z^{2i}\frac{1}{2}(S_{2i-1}+1) + \sum_{i=1}^{\infty} z^{2i-1}\frac{1}{2}(1-S_{2i-2}) \tag{5.2}$$

It was shown in Section 3 [compare Eqs. (3.13) and (3.14)] that the characteristic polynomial for parameter values $a<\sqrt{2}$ can be written in the form

$$0=\sum_{i=0}^{\infty}\sigma_{2i}z^{2i}=-1+\sum_{i=0}^{\infty}S_{2i}z^{2+2i} \tag{5.3}$$

and

$$S_{2i}=-S_{2i+1} \tag{5.4}$$

Inserting Eqs. (5.4) and (5.3) into Eq. (5.2) and evaluating the geometrical series, we finally get

$$P_{a,\lambda}(x)\,|_{[f^2(0),f(0)]}=0 \tag{5.5}$$

The critical trajectory $f^{n+2}(x_c)=f^n(0)$ is attracted to the fixed point $x=a/(a+1)$ if $a=\sqrt{2}$. If a is smaller, the critical trajectory misses it and $f^2(0)$ and $f(0)$ define the left and right boundaries of two disjoint intervals $I_2=[0, f^2(0)]$ and $I_1=[f(0), 1]$, respectively. The invariant density has become positive semidefinite and is split into two positive definite bands. All other eigenfunctions are likewise confined to the two invariant intervals newly brought into existence. The indicator functions of the intervals I_1 and I_2 will be denoted χ_1 and χ_2, respectively. We can simplify the expressions for the eigenfunctions in I_1 and I_2 again making use of Eqs. (5.3) and (5.4):

$$P_{a,\lambda}(x)\,\chi_2 = \left\{\frac{1}{2}\frac{1}{z}\frac{1-2z^2}{1-z^2} + \sum_{i=1}^{\infty} z^{2i-1}\frac{1}{2}(1-S_{2i-2}) + \sum_{i=1}^{\infty} z^{2i}\theta(S_{2i-1}[x-f^{2i}(0)])\right\}\chi_2 \tag{5.6}$$

Taking into account that $\theta(s_1[x-f^2(0)])$ contributes to the constant, we obtain

$$P_{a,\lambda}(x)\,\chi_2 = \left\{\frac{1}{2}\frac{1}{z^2}\frac{1-2z^4}{1-z^2} + \sum_{i=1}^{\infty} z^{2i}\theta(s_{2i+1}[x-f^{2i+2}(0)])\right\} z^2\chi_2 \tag{5.7}$$

and the interval I_1:

$$P_{a,\lambda}(x)\,\chi_1 = \left\{\frac{1}{2}\frac{1}{z^2}\frac{1-2z^4}{1-z^2} + \sum_{i=1}^{\infty} z^{2i}\theta(S_{2i}[x-f^{2i+1}(0)]\right\} z\chi_1 \tag{5.8}$$

It is easy to find a conjugacy, which connects the eigenfunctions confined in the bands I_1 and I_2 with eigenfunctions nonzero on the whole interval and control parameter a^2: Define the transformation

$$\begin{aligned} z_2 &\to z' \\ S_{2i-2} &\to S'_{i-1} \end{aligned} \tag{5.9}$$

and

$$T_{1,a}\colon x = f_a(0) + [1-f_a(0)]\,x' \tag{5.10}$$

in the interval I_1, mapping $[f_a(0), 1] \to [0, 1]$ and

$$T_{2,a}\colon x = f_a^2(0) - f_a^2(0)\,x' \tag{5.11}$$

in the interval I_2, mapping $[0, f^2(0)] \to [1, 0]$. Both conjugacies transform the map $x_{n+1} = f_a(x_n)$ into the map $x'_{n+1} = f_{a^2}(x'_n)$. Furthermore, we find two identities

$$f_{a^2}(x) = \frac{1}{1-f_a(0)}\{f_a^2[f_a(0) + x(1-f_a(0))] + f_a(0)\} \tag{5.12}$$

and

$$f_{a^2}(x) = 1 - \frac{1}{f_a^2(0)} f_a^2[f_a^2(0) - f_a^2(0)\,x] \tag{5.13}$$

Making use of Eqs. (5.10)–(5.14), we obtain

$$P_{a,\lambda}(\{S_i\}, x) = \frac{1}{\lambda a}\chi_1 P_{a^2,\lambda^2}(\{S_{2i}\}, T_{1,a}^{-1}(x))$$

$$+ \frac{1}{\lambda^2 a^2}\chi_2 P_{a^2,\lambda^2}(\{S_{2i}\}, T_{2,a}^{-1}(x)) \qquad (5.14)$$

where we have indicated the corresponding symbol sequences explicitly. Thus, the eigenfunctions of the different bands are similar, except that the orientation in band 2 is reversed owing to the negative Jacobian of Eq. (5.12). Furthermore, there are two linear conjugacies which transform the eigenfunction $P_{a,\lambda}(x)$ into eigenfunctions $P_{a^2,\lambda^2}(x)$.

If $a^2 = \sqrt{2}$, the next band splitting occurs within the two bands, and we can repeat the whole procedure for the renormalized expression (5.14) in each band separately. A new band splitting will occur at all points $a = 2^{2^{-N}}$ and we get the eigenfunctions

$$P_{a,\lambda}(\{S_i\}, X) = \frac{\|D\tau_0\|}{2^N}\sum_{k=0}^{2^N-1}\left(\frac{1}{\lambda a}\right)^k \chi_k P_{a^{2^N},\lambda^{2^N}}(\{S_{2^N i}\}, \tau_k(x)) \qquad (5.15)$$

in this expression, the bands are ordered such that

$$f\chi_k = \chi_{k+1}, \qquad k \in \mathbb{Z}_{2^N} \qquad (5.16)$$

where χ_0 is the indicator function of the interval $I_0 = [f^{2^N-1}(0), 1]$ and I_{2^N-1} is the preimage of I_0 and contains the critical point. The linear transformation τ_k maps the unit interval onto the kth band, taking into account the correct orientation. It is a product of N inverse transformations $T_{1,a}^{-1}$ or $T_{2,a}^{-1}$, respectively. The order within the product depends on the kneading invariant. The Jacobian of τ_k is

$$\|D\tau_k\| = \left(\frac{1}{a}\right)^k \|D\tau_0\| \qquad (5.17)$$

because the map is everywhere linear and I_0 is the shortest interval. The norm is chosen such that

$$\|D\tau_k\| \int_{I_k} P_{a^{2^N},1}(\{S_{2^N i}\}, \tau_k(x))\,\chi_k\,dx = 1 \qquad (5.18)$$

and the prefactor 2^{-N} ensures that

$$\int_0^1 P_{a,1}(x)\,dx = 1 \qquad (5.19)$$

A set of left eigenfunctions of H, which belongs to the eigenvalues $\lambda_r = \exp(2\pi i r \cdot 2^{-N})$, $r \in \mathbb{Z}_{2^N}$, is given by

$$P^L_{a,\lambda}(\{S_i\}, x) = \sum_{k=0}^{2^N-1} \lambda^k \chi_k^{(N)} \tag{5.20}$$

The scalar product reads [$P^R = P$; compare Eq. (5.15)]

$$\int_0^1 P^L_{a,\lambda}(x)\, P^R_{a,\lambda'}(x)\, dx = \sum_{k=0}^{2^N-1} \left(\frac{\lambda}{\lambda'}\right)^n \frac{1}{2} N \int_0^1 P_{a^{2^N},\lambda^{2^N}}(x)\, dx \tag{5.21}$$

If $\lambda^{2^N} \neq 1$, the right-hand side of Eq. (5.21) vanishes because of the conservation of the norm. If $\lambda' = \exp(2\pi i r' \cdot 2^{-N})$, we get

$$\int_0^1 P^L_{a,\lambda_r} \cdot P^R_{a,\lambda_{r'}}\, dx = 2^{-N} \sum_{k=1}^{2^N-1} e^{2\pi i(r-r')k2^{-N}} = \delta_{rr'} \tag{5.22}$$

The left eigenfunctions are eigenfunctions of the adjoint of the Frobenius–Perron operator H:

$$H^+ P^L(x) \equiv P^L(f(x)) = \lambda P^L(x) \tag{5.23}$$

It is easy to convince oneself that the only eigenfunctions of H^+ have eigenvalues with modulus 1: Take the absolute of Eq. (5.23) and integrate it with respect to the invariant density $\mu(x)$,

$$\int_0^1 \mu(x)\, |P^L(f(x))|\, dx = |\lambda| \int_0^1 \mu(x)\, |P^L(x)|\, dx \tag{5.24}$$

Using the invariance of the left-hand side of eq. (5.24) under the transformation $x' = f(x)$, it follows that $|\lambda| = 1$. Thus, the functions $\{P^L_{a,\lambda}(x)\}$, we found above, are the only left eigenfunctions we can expect to find. There are no left eigenfunctions which enable us to project an arbitrary initial distribution to eigenfunction with eigenvalues λ of modulus smaller than 1.

If, on the other hand, the critical trajectory is periodic and we only allow for piecewise constant functions with jumps at $\{f^n(x_c\}$, then H can be written as a finite-dimensional matrix (H_{nm}) and a left eigenbasis is easily constructed. But although there is a unique map from the eigenvectors of (H_{nm}) to eigenfunctions of H, there is no such map for the left set and results from the finite-dimensional formalism cannot be used for the general problem.

6. THE ASYMPTOTIC EXPANSION OF INITIAL DISTRIBUTION OF BOUNDED VARIATION

Although there is no hope of finding an appropriate set of left eigenfunctions which allows us to express an arbitrary initial probability density in terms of an infinite sum of eigenfunctions, there are projectors (but no eigenfunctions) which make an asymptotic expansion possible. In view of what was said at the end of Section 4 it is only natural that this algorithm applies only for the restricted class of functions $h(x) \in L_2(\mu)|_{BV}$. Because $h(x)$ is a probability density a change of $h(x)$ on a set of measure zero does not change the final results on measures and correlation functions. Therefore we can restrict $h(x)$ further:

(1) $h(0)=0$ but usually $h(0^+) \neq 0$

(2) $h(x)$ continuous for all $x \in Y$

where $Y=\{x \mid f^n(x)=x_c \text{ and } n \in \mathbb{N}\}$, i.e., $h(x)$ is continuous on all preimages of x_c. Because $h(x)$ is of bounded variation it could have jumps only on a countable subset of Y. Thus, the change of $h(x)$ by shifting a jump on $x \in Y$ to a neighboring $x \notin Y$ can be made to be of measure zero.

Using Eq. (2.16) the relevant eigenfunctions [Eq. (2.18)] can be transformed into a more convenient form:

$$P_n(x)=\sum_{k=0}^{\infty} \sigma_{k+1}(\lambda_n)^k \theta(x-f^k(0)) \tag{6.1}$$

After these remarks we can state the basic facts:

Theorem. Let $h(x) \in L_2(\mu)|_{BV}$ such that $h(0)=0$ and $h(x)$ continuous on Y. Let $P_n(x)$ be a relevant eigenfunction with eigenvalue λ_n. Then there is a function $\Omega(x, z_m)$, $z_m=1/(a\lambda_m)$ defined by

$$\Omega(x, z_m)=\omega(x, z_m) \operatorname{Res} 1/\rho(z_m) \tag{6.2}$$

where $\omega(x, z_m)=\sum_{k=0}^{\infty} (z_m)^k \operatorname{sign}[(d/dx) f^k(x)]$.
such that the following relations hold (the integration is understood in the sense of Riemann–Stieltjes):

(i) $$\int_0^1 d(P_n(x))\, \Omega(x, z_m)=\delta_{nm} \tag{6.3}$$

(ii) $$\int_0^1 d(H^n h(x))\, \Omega(x, z_m)=(\lambda_m)^n \int_0^1 d(h(x))\, \Omega(x, z_m) \tag{6.4}$$

(iii) Let $n_n(x) \in N_n \cap L_2(\mu)|_{BV}$, then

$$\int_0^1 d(n_n(x))\, \Omega(x, z_m)=0 \tag{6.5}$$

$$\text{(iv)} \quad \left\| H^n h(x) - \sum_{m=0}^{M} \alpha_m (\lambda_m)^n P_m(x) \right\|_2 \leqslant c\eta^n \tag{6.6}$$

where $1/a < \eta < |\lambda_M|$, λ_i are the $M+1$ relevant eigenvalues with largest modulus and the coefficients α_m are determined by the integral

$$\alpha_m = \int_0^1 d(h(x))\, \Omega(x, z_m) \tag{6.7}$$

The existence of integrals of type $\int_0^1 d(h(x))\, \Omega(x, z)$ is guaranteed if $\omega(x, z)$ is continuous with respect to x. Defining

$$\sigma_k(x) = \text{sign} \frac{d}{dx} f^k(x) \tag{6.8}$$

$\omega(x, z)$ can be written as

$$\omega(x, z) = \sum_{k=0}^{n} (z)^k \sigma_k(x) + O(z^{n+1}) \tag{6.9}$$

Let X_k be the set of all preimages of order k of x_c

$$X_k := \{x \in [0, 1] \mid f^k(x) = x_c\} \tag{6.10}$$

$Y_k = \bigcup_{n=0}^{k} X_n$ and $Y = Y_\infty$. Now suppose, there is exactly one element $x_0 \in [x_1, x_2]$ which is also an element of X_l. This implies, that $\sigma_k(x_1) = \sigma_k(x_2)$ if $k = 0, ..., l-1$ and $\sigma_k(x_1) = -\sigma_k(x_2)$ if $k = l, ..., n$. Taking the special value $z = z_m$ and using $\rho(z_m) = 0$ we find

$$\sum_{k=l}^{n} \sigma_k(x_2)(z_m)^k = -(z_m)^l \sum_{k=n-l}^{\infty} \sigma_k(z_m)^k \tag{6.11}$$

where we have used, that $\sigma_{l+k}(x_2) = \sigma_k(x_c) \equiv \sigma_k (k \leqslant n-l)$. Thus

$$|\omega(x_1, z_m) - \omega(x_2, z_m)| \leqslant c\, |z_m|^{n+1} \tag{6.12}$$

where $C \leqslant 4/(1-|z|)$. For all $\varepsilon > 0$ there is an n such that $c|z_m|^{n+1} \geqslant \varepsilon > c\, |z_m|^n$. The exponent n fixes the set Y_n on which $\omega(x, z_m)$ could have discontinuities. But Y_n has only a finite number of elements. Thus there is a δ such that for all x the set $[x - \delta/2, x + \delta/2] \cap Y_n$ contains at most one element. After having fixed δ in this way inequality (6.12) is guaranteed for all $|x_1 - x_2| < \delta$. Thus $\omega(x, z_m)$ is continuous.

For simplicity we will assume that all zeros of $\rho(z)$ we are concerned with are simple. In order to prove (i), we start with the trivial fact that

$$\lim_{z \to z_n} \text{Res}(1/\rho(z_m))\, \rho(z)/(z - z_m) = \delta_{nm} \tag{6.13}$$

where z_m is a simple zero of $\rho(z)$. $\rho(z)$ is analytic, thus we can write

$$\frac{\rho(z)}{z-z_m}=\sum_{i=0}^{\infty}(z)^i\frac{1}{i!}\left[\left(\frac{d}{dz}\right)^i\frac{\rho(z)}{z-z_m}\bigg|_{z=0}\right]$$

$$=\sum_{i=0}^{\infty}(z)^i\sum_{j=0}^{\infty}\sigma_{i+1+j}(z_m)^j$$

$$=\sum_{i=0}^{\infty}(z)^i\sigma_{i+1}\sum_{j=0}^{\infty}(z_m)^j\operatorname{sign}\frac{d}{dx}f^j(x)\,|_{x=f^i(0)} \tag{6.14}$$

In the limit $z\to z_n$, we finally get

$$\lim_{z\to z_n}\frac{\rho(z)}{z-z_m}=\int_0^1 d(P_n(x))\,\omega(x,z_m) \tag{6.15}$$

The proof of (ii) is purely calculational. We can restrict ourselves to the case $n=1$, all other n follow accordingly.

$$\int_0^1 d(h(x))\sum_{n=0}^{\infty}\left[\operatorname{sign}\frac{d}{dx}f^n(x)\right](z_k)^n$$

$$=h(1)+\int_0^1 d(h(x))\sum_{n=1}^{\infty}\left[\operatorname{sign}\frac{d}{dx}f^n(x)\right](z_k)^n \tag{6.16}$$

Introduce the substitution $x'=f(x)$

$$h(1)+\int_0^1 d(hf_+^{-1}(x'))\sum_{n=1}^{\infty}\operatorname{sign}\frac{d}{dx'}f^{n-1}(x')(z_k)^n$$

$$+\int_0^1 d(hf_-^{-1}(x'))\sum_{n=1}^{\infty}\operatorname{sign}\frac{d}{dx'}f^{n-1}(x')(z_k)^n$$

$$=h(1)+1/\lambda_k\int_0^1 d(Hh(x'))\sum_{n=0}^{\infty}\operatorname{sign}\frac{d}{dx'}f^n(x')(z_k)^n \tag{6.17}$$

The term $h(1)$ can be absorbed into the integral modifying the lower boundary of $Hh(x)$ into

$$Hh(0)=0 \qquad \text{but } Hh(0^+)\neq 0$$

in accordance with the restrictions we imposed on the function space. From this, proposition (ii) follows immediately.

Now, take any function $n_l(k)$ of bounded variation from the null space N_l and apply the result we found above $l+1$ times. Then we find

$$(\lambda_k)^{l+1}\int_0^1 d(n_l(x))\,\omega(x, z_k)=\int_0^1 d(H^{l+1}n_l(x))\,\omega(x, z_k)=0 \qquad (6.18)$$

by the definition of N_l. This proves part (iii) of the theorem.

In order to prove the fourth conjecture we investigate at first the nth iterate of $h(x)$:

$$H^n h(x)=\frac{1}{a^n}\sum_{i=0}^{p-1}\left[\operatorname{sign}\frac{d}{dx}f^n(\xi_i)\right](\theta(x-f^n(\xi_i)) - \theta(x-f^n(\xi_{i+1})))\,h(f_i^{-n}(x)) \qquad (6.19)$$

where the sum is taken over all laps of $f^n(x)$ and $\xi_i \in Y_n$. This can be written as

$$H^n h(x)=2a^{-n}\sum_{i=0}^{p-1}\left[\operatorname{sign}\frac{d}{dx}f^n(\xi_i)\right]\theta(x-f^n(\xi_i))\,h(\xi_i) + C_n(x)\,2a^{-n}\operatorname{Var}(h(x)) \qquad (6.20)$$

where $|C_n(x)|\leqslant 1$ for all n. Ordering the sequence $\{\xi_i\}$ according to different orders of preimages of x_c, Eq. (6.20) becomes

$$H^n h(x)=2a^{-n}\sum_{m=0}^{n-2}\sigma_{m+1}\,\theta(x-f^m(0))\sum_{x\in X_{n-m-2}}h(x) + C_n(x)\,2a^{-n}\operatorname{Var}(h(x)) \qquad (6.21)$$

We see from this expression, that any $h(x)\in L_2(\mu)\,|_{BV}$ decays exponentially to $L\theta(\mu)$.

Approximating $h(x)$ by a finite number of relevant eigenfunctions $P_l(x)$, we find

$$H^n\bar{h}(x)=\sum_{l=0}^{M}(\lambda_l)^n\,2\pi i\,P_l(x)\oint_{c_l}\frac{dz}{\rho(z)} \times\int_0^1 d(h(x'))\sum_{k=1}^{\infty}\operatorname{sign}\frac{d}{dx'}f^k(x')(z_l)^k \qquad (6.22)$$

Where $\bar{h}(x)$ is an approximation of $h(x)$ and we assume for simplicity $h(0^+)=0$. The error created by this approximation is not larger than

$h(0^+)\, a^{-n}$. C_l is a tiny circle around $z = z_l$ in the complex plane. Using the definition of $P_l(x)$ and reordering the sum, we get

$$\sum_{m=0}^{\infty} \sigma_{m+1} \theta(x - f^m(0))\, a^{-n} \int_0^1 d(h(x)) \sum_{k=1}^{\infty} \left[\operatorname{sign} \frac{d}{dx} f^k(x) \right] \times \sum_{l=0}^{M} 2\pi i \oint_{C_l} \frac{z^{m-n+k}}{\rho(z)}\, dz \tag{6.23}$$

For the next step, it is essential that $\{|z_0|, ..., |z_M|\}$ are the $M+1$ smallest zeros of $\rho(z)$. Then the sum over the residues of $1/\rho(z)$ can be transformed into

$$\sum_{l=0}^{M} \oint_{C_l} \frac{z^p}{\rho(z)}\, dz = -\oint_{\gamma} \frac{z^p}{\rho(z)}\, dz + \oint_{\Gamma} \frac{z^p}{\rho(z)}\, dz \tag{6.24}$$

where γ is a small circle around $z = 0$ and Γ is a large circle of radius r enclosing exactly the smallest $M+1$ zeros of $\rho(z)$. The contribution of the second integral can be estimated

$$\left| \sum_{m=0}^{\infty} \sigma_{m+1} \theta(x - f^m(0))\, a^{-n} \int_0^1 d(h(x)) \sum_{k=1}^{\infty} \operatorname{sign} \frac{d}{dx} f^k(x)\, 2\Pi i \oint_{\Gamma} \frac{z^{m-n+k}}{\rho(z)}\, dz \right|$$

$$\leqslant \sum_{m=0}^{\infty} a^{-n} \int_0^1 |d(h(x))| \sum_{k=1}^{\infty} 2\pi r^{m-n+k} 2\Pi \int_0^{2\pi} \frac{d\varphi}{|\rho(z)|} \tag{6.25}$$

$$= (ar)^{-n} \frac{r}{(1-r)^2} \operatorname{Var}(h(x)) \int_0^1 1/|\rho(re^{2\Pi i\varphi})|\, d\varphi \tag{6.26}$$

But we know, that $|\lambda_{M+1}| < 1/(ra) < |\lambda_M|$. Then there is a uniformally bounded function $C'(x)$, $|C'(x)| \leqslant 1$ such that expression (6.25) is equal to

$$C'(x)\, \eta^n \frac{r}{(1-r)^2} \operatorname{Var}(h(x)) \int_0^1 \frac{d\varphi}{|\rho(z(\varphi))|} \tag{6.27}$$

where $|\lambda_{M+1}| < \eta < |\lambda_m|$.

Evaluating the integral around $z = 0$, we note that

$$\oint_{\gamma} z^{m-n+k}/\rho(z)\, dz \neq 0$$

only if $k \leqslant n - m - 1$ because $\rho(z)$ is analytic at the origin. So m has to be smaller than $n-1$ and the largest possible k is $n-m-1$. Then all sums in expression (6.23) are finite and we have to evaluate

$$\sum_{m=0}^{n-2} \sigma_{m+1}\theta(x-f^m(0))\, a^{-n} \int_0^1 d(h(x')) \sum_{k=1}^{n-m-1} \operatorname{sign} \frac{d}{dx'} f^k(x')(-2\pi i) \times \oint_\gamma \frac{z^{m-n+k}}{\rho(z)}\, dz \tag{6.28}$$

Having only sums of finite length, we can interchange integration and summation:

$$a^{-n} \sum_{k=1}^{n-m-1} (-2\pi i) \oint_\gamma \frac{z^{m-n+k}}{\rho(z)}\, dz \int_0^1 d(h(x)) \operatorname{sign} \frac{d}{dx} f^k(x) \tag{6.29}$$

The Riemann–Stieltjes integral can be calculated

$$\int_0^1 d(h(x)) \operatorname{sign} \frac{d}{dx} f^k(x) = -2 \sum_{p=0}^{k-1} \sigma_{k-p-1} \sum_{x \in X_p} h(x) \tag{6.30}$$

Inserting this expression into Eq. (6.29) and rearranging the sums we arrive at

$$a^{-n} \sum_{p=0}^{n-m-2} (4\pi i) \oint_\gamma z^{m-n+1+p} \rho_{n-m-2-p}(z)/\rho(z)\, dz \sum_{x \in X_p} h(x) \tag{6.31}$$

where $\rho_r(z) = \sum_{l=0}^{r} \sigma_l(z)^l$. Now we can write

$$\rho_r(z) = \rho(z) - z^{r+1}\tilde{\rho}(z) \tag{6.32}$$

and $\tilde{\rho}(z)$ is an analytic function. Then the integral around $z=0$ becomes

$$\oint_\gamma z^{m-n+1+p}\, dz - \oint_\gamma \tilde{\rho}(z)/\rho(z)\, dz = 1/(2\pi i)\, \delta_{p,n-m-2} \tag{6.33}$$

So, expression (6.28) is reduced to

$$\sum_{m=0}^{n-2} \sigma_{m+1}\theta(x-f^m(0))\, 2a^{-n} \sum_{x \in X_{n-m-2}} h(x) \tag{6.34}$$

Putting together expressions (6.20), (6.22), (6.27), and (6.34) we finally get

$$\|H^n h(x) - H^n \bar{h}(x)\|_2 \leqslant \eta^n \cdot C \cdot \operatorname{Var}(h(x))$$

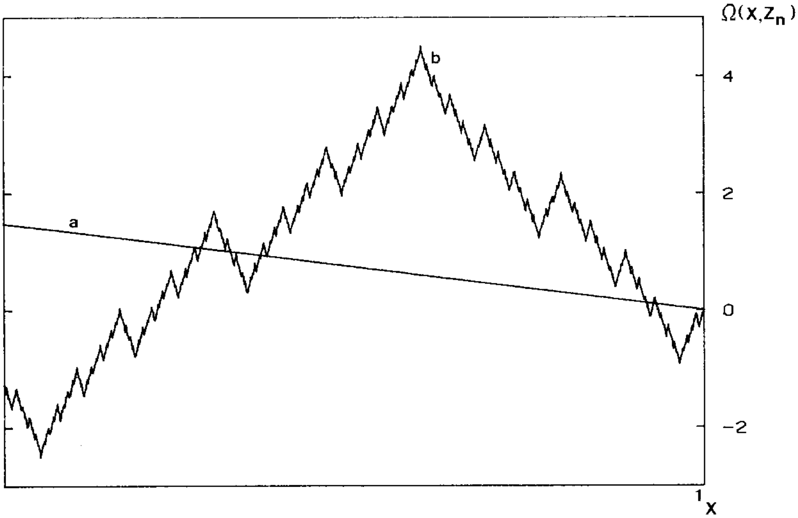

Fig. 4. The projecting functions $\Omega(x, z_m)$ are calculated $(a = 1.48)$. (a) belongs to the invariant density $(z_0 = 1/a)$, (b) belongs to the eigenfunction with eigenvalue $\lambda_1 = -0.86723$ $(z_1 = -1/\lambda_1 a)$.

where $1/a < \eta < |\lambda_M|$ and

$$C = \frac{r}{1-r^2} \int_0^1 \frac{d\varphi}{|\rho|} + 2 \quad \blacksquare$$

It is possible to find an analytic expression for the function $\omega(x, z_l)$ provided $|z_l| = 1/a$, i.e., $|\lambda_l| = 1$. From the expansion of a number $x \in (0, 1)$ into its λ series,[6] one easily deduces that for all a

$$\omega(x, 1/a) = a(1-x) \tag{6.36}$$

If a is smaller than $\sqrt{2}$, i.e., after the first band splitting has occurred, the restrictions on the kneading invariant yield another functional relation:

$$\begin{aligned} &\frac{\omega(x, z) - 1}{z-1} - \frac{\omega(x, -z) - 1}{z+1} = 0, && x < x^* \\ &\frac{\omega(x, z)}{z-1} + \frac{\omega(x, -z)}{z+1} = 0, && x \geqslant x^* \end{aligned} \tag{6.37}$$

where $f(x^*) = x^*$. This relation is valid for all z. Using the expansion (6.36) one easily gets

$$\omega(x, -1/a) = \begin{cases} \dfrac{2a}{a-1} - \dfrac{a+1}{a-1}\, a(1-x), & x < x^* \\[2ex] \dfrac{a+1}{a-1}\, a(1-x), & x \geqslant x^* \end{cases} \tag{6.38}$$

Similar relations hold at all band-splitting points, so that $\omega(x, z)$ can be expressed in terms of $\omega(x, 1/a)$ provided $|z_l| = 1$.

Integrating expansion (6.6) by parts, we obtain

$$\int_0^1 d(h(x))\, \theta(x, z_l) = -\int_0^1 h(x)\, d\Omega(x, z_l) \tag{6.39}$$

[$h(0) = 0$ by definition and $\Omega(1, z_l) = 0$ because $\operatorname{sign}(d/dx) f^n(x)\,|_{x=1} = -\sigma_n$]. The existence of the left-hand side implies the existence of the right-hand side. But if $|z_l| = 1$, then $\Omega(x, z_l)$ is a piecewise linear function and we can write

$$\int_0^1 d(h(x))\, \Omega(x, z_l) = \int_0^1 P_l^L(x)\, h(x)\, dx \tag{6.40}$$

where $P_l^L(x) = (d/dx)\, \Omega(x, z_l)$ are the left eigenfunctions of H with eigenvalue $\lambda_l (|\lambda_l| = 1)$ we already discussed in the previous section.

As an application of this algorithm we can calculate the asymptotic behavior of the correlation function $\langle x_n x \rangle$:

$$\begin{aligned} \langle x_n x \rangle &= \int_0^1 dx\, x H^n(x\mu(x)) \\ &\quad \times \sum_{l=0}^{M} \int_0^1 d(x\mu(x))\, \Omega(x, z_l)(\lambda_l)^n \int_0^1 dx x \sum_{k=0}^{\infty} \sigma_{k+1}(z_l)^k\, \theta(x - f^k(0)) \\ &= \frac{1}{2} \sum_{l=0}^{M} (\lambda_l)^n \int_0^1 d(x\mu(x))\, \Omega(x, z_l) \left\{ z_l - \sum_{k=0}^{\infty} \sigma_{k+1}(z_l)^k [f^k(0)]^2 \right\} \end{aligned} \tag{6.41}$$

where the sum over l includes the $M+1$ relevant eigenvalues with largest moduli. It follows, that the correlation function $\langle x_n x \rangle$ decays exponentially to its stationary part. A similar calculation for an arbitrary correlation function $\langle g_1(x_n)\, g_2(x) \rangle$ leads qualitatively to the same results provided $g_1(x)$ and $g_2(x)$ are of bounded variation. In the following section, the nondecaying part of Eq. (6.41) will be further evaluated.

7. THE STATIONARY BEHAVIOR OF THE CORRELATION FUNCTION $\langle x(n)x \rangle$

Let's suppose $2^M < a < 2^{M/2}$ where $M = 2^{-N}$, i.e., we have 2^N disjoint bands. The 2^N left eigenfunctions with eigenvalues $|\lambda_r| = 1$ can be written formally in the form

$$P^L_{a,\lambda_r}(x) = P^{R*}_{a,\lambda_r}(x)/P^R_{a,1}(x) \tag{7.1}$$

($P^L := 0$, if the right-hand side of Eq. (7.1) is of the form 0/0). The Jacobian

$$\|D\tau_0\|^{-1} = l_0^{(N)} = \prod_{\kappa=0}^{N-1} (a^{2^\kappa} - 1) \tag{7.2}$$

which is equal to the width $l_0(N)$ of this band. The itinerary of the critical trajectory sweeps the boundaries of all intervals during the first 2^{N+1} iterations and enters the interior of I_0 after $2^{N+1} + 1$ iterations. Thus, the interval I_k is given by

$$I_k = [f^{k+1}(x_c), f^{k+1+2^N}(x_c)] \tag{7.3}$$

and the orientation of an interval I_k is determined by the element $-\sigma_\kappa$ of the kneading invariant. The invariant density in the interval I_0 does not change its orientation from band splitting to band splitting, whereas the orientation of the last interval (i.e., the interval which contains the critical point) is reversed each time a new set of bands comes into existence.

In the following, we will concentrate on the correlation function $\langle x(n)\, x \rangle$. It can be written as

$$\langle x(n)\, x \rangle = \int_I x \cdot H^n(x\mu(x)\, dx) \tag{7.4}$$

where $\mu(x)$ is the invariant density $P_{a,1}(x)$. We want to expand $xP_{a,1}$ in terms of P_{a,λ_r}:

$$xP_0 = \sum_{p=0}^{2^N-1} \alpha_p P_p \tag{7.5}$$

$[P_p \equiv P_{a,\lambda_p}(x)]$. From Eq. (7.1) and Eq. (5.22) follows that $\alpha_p = \langle x \rangle_p^*$, where $\langle x \rangle_p = \int_I xP_p(x)\, dx$.

Thus we get

$$x\mu = \sum_p \langle x \rangle_p^* P_p \tag{7.6}$$

and the correlation function $\langle x(n)\, x\rangle$ becomes

$$\langle x(n)\, x\rangle = \sum_{p=0}^{2^N-1} (\lambda_p)^n\, |\langle x\rangle_p|^2 = \sum_{p=0}^{2^N-1} e^{2\pi inp2^{-N}}\, |A_N(p)|^2 \tag{7.7}$$

where $|A_N(p)|^2$ is the power spectrum of the correlation function:

$$A_N(p) = 2^{-N} \sum_{k=0}^{2^N-1} l^{-2\pi ikp2^{-N}} \int_{I_k} |D\tau_k|\, P_{a^{2^N},1}(\tau_k(x))\, x\, dx \tag{7.8}$$

The integrals in Eq. (7.8) can be rewritten

$$\int_{I_k} |D\tau_k|\, P_{a^{2^N},1}(\tau_k(x))\, x\, dx = f^{k+1}(x_c) + \{f^{k+1}(x_c) - f^{2^N+k+1}(x_c)\}\, \eta \tag{7.9}$$

and

$$\eta = \int_0^1 P_{a^{2^N},1}(\xi)(\xi - 1)\, d\xi \tag{7.10}$$

We can split A_N in a deterministic part $A_N^{(d)}$ and an intraband part $A_N^{(\eta)}$, containing η. $A_N^{(\eta)}$ can be evaluated straightforwardly, exploiting that

$$f^{k+1}(x_c) - f^{2^N+k+1}(x_c) = -\sigma_k l_0^{(N)} a^k \tag{7.11}$$

where $l_0(N)$ is the width of the smallest band.

Then $A_N^{(\eta)}(p)$ can be expressed as

$$A_N^{(\eta)}(p) = -2^{-N}\eta l_0^{(N)} \sum_{k=0}^{2^N-1} e^{-2\pi ipk2^{-N}} \sigma_k a^k = \eta 2^{-N} l_0^{(N)} \prod_{\kappa=0}^{N-1} \phi_{N,\kappa}(p) \tag{7.12}$$

with

$$\phi_{N,\kappa}(p) = (1 - a^{2^\kappa} e^{-2\pi ip^{-N+\kappa}}) \tag{7.13}$$

$\phi_{N,\kappa}(p)$ has the property, that

$$\phi_{N,N-1}(p) = a^{2^{N-1}} + 1 \tag{7.14}$$

and

$$\phi_{N,N+v}(p) = -\alpha_N \qquad \text{where } \alpha_\kappa \equiv a^{2^\kappa} - 1 \tag{7.15}$$

for all $v > 0$. The deterministic part $A_N^{(d)}(p)$ can also be evaluated exactly. If $p = 2^\mu \cdot q$, where q is any odd number such that $p < 2^N$, we get

$$\begin{aligned} A_N^{(d)}(2^\mu \cdot q) &= \frac{1}{2^N} \sum_{k=0}^{2^N - 1} e^{-2\pi i k q 2^{\mu - N}} f^{k+1}(x_c) \\ &= \frac{1}{2^{-N+1}} \sum_{k=0}^{2^{N-1}-1} e^{-2\pi i k q 2^{\mu-N}} f^{k+1}(x_c) \\ &\quad - \frac{1}{2^{-N}} \sum_{k=0}^{2^{N-1}-1} e^{-2\pi i k q 2^{\mu-N}} [f^{k+1}(x_c) - f^{k+1+2^{N-1}}(x_c)] \end{aligned} \tag{7.16}$$

The first part of the right-hand side of Eq. (7.16) is $A_{N-1}^{(d)}(2^{\mu-1}q)$. The procedure can be repeated μ times:

$$\begin{aligned} A_N^{(d)}(2^\mu \cdot q) = &- \sum_{i=1}^{\mu} 2^{-N+i-1} l_0^{(N-i)} \left\{ - \sum_{k=0}^{2^{N-i}-1} e^{-2\pi i q k 2^{-N+\mu}} a^k \sigma_k \right\} \\ &+ 2^{-N+\mu} \left\{ - \sum_{k=0}^{2^{N-\mu-1}-1} l_0^{(N-\mu-1)} \sigma_k a^k e^{-2\pi i q k 2^{-N+\mu}} \right\} \end{aligned} \tag{7.17}$$

The sums over k can be performed exactly and we obtain the power spectrum:

$$\begin{aligned} A_N(2^\mu \cdot q) = &- \sum_{i=1}^{\mu} 2^{-N+i-1} l_0^{(N-i)} \prod_{\kappa=0}^{N-i-1} \phi_{N-\mu,\kappa}(q) \\ &+ 2^{-N+\mu} l_0^{(N-\mu+1)} \prod_{\kappa=0}^{N-\mu-2} \phi_{N-\mu,\kappa}(q) \\ &+ \eta 2^{-N} l_0^{(N)} \prod_{\kappa=0}^{N-1} \phi_{N-\mu,\kappa}(q) \end{aligned} \tag{7.18}$$

Expression (7.18) can be evaluated further, distinguishing three different cases:

(a) $\mu = N$,

$$A_N(2^N) = \frac{1}{2} - \frac{1}{4} \sum_{i=0}^{N-2} 2^{-i}(-1)^i \prod_{\kappa=0}^{i} \alpha_\kappa^2 + \eta 2^{-N}(-1)^N \prod_{\kappa=0}^{N-1} \alpha_\kappa^2 \tag{7.19}$$

(b) $\mu = N - 1$,

$$A_N(2^{N-1}) = \frac{1}{2} - \frac{1}{4} \alpha_1 \sum_{i=0}^{N-2} 2^{-i}(-1)^i \prod_{\kappa=1}^{i} \alpha_\kappa^2 + \eta 2^{-N}(-1)^{N-1} \alpha_1 \prod_{\kappa=1}^{N-1} \alpha_\kappa^2 \tag{7.20}$$

(c) $\mu < N-2$,

$$A_N(2^\mu q) = 2^{-N+\mu} \prod_{\kappa=0}^{N-\mu-2} \alpha_\kappa \phi_{N-\mu,\kappa}(q)$$
$$\times \left\{1 - \frac{1}{2}\alpha_{N-\mu} + \frac{1}{4}\alpha_{N-\mu} \sum_{i=0}^{\mu-2} 2^{-i}(-1)^i\right\}$$
$$\prod_{\kappa=0}^{i} \alpha^2_{N-\mu+\kappa} + \eta 2^{-\mu}\alpha_{N-\mu}(-1)^\mu \prod_{\kappa=0}^{\mu-1} \alpha^2_{N-\mu+\kappa}\Bigg\} \tag{7.21}$$

Equations (7.19)–(7.21) give an exact expression for the power spectrum of the nondecaying part of the correlation function $\langle x(n)\, x\rangle$. With each new band splitting $N2^{N-1}$ new spectral lines come into existence at the points $2\pi q \cdot 2^{-N}$. Equation (7.19) determines the amplitude of the basic frequency $v_0 = \pi$, Eq. (7.20) that of the first subharmonic $v_1 = \pi/2$, and Eq. (7.21) those of all subsequent subharmonics $vq = \pi \cdot q \cdot 2^{-M}$. The intensity of the subharmonic frequencies last to come into existence is obtained from Eq. (7.21), setting $\mu = 0$

$$A_N(q) = 2^{-N}\{1 + \eta\alpha_N\} \prod_{\kappa=0}^{N-2} \alpha_\kappa \phi_{N,\kappa}(p) \tag{7.22}$$

Concentrating ourselves on some particular line of the spectrum (i.e., $N - \mu = M = \text{const}$, but N and μ increasing) we observe from Eq. (7.21) that the contribution to the correlation function from the intraband mixing vanishes at least exponentially with increasing μ and N. Furthermore, the products $\prod \alpha_\kappa^2$ which are connected with the width of different bands become extremely small after a few iterations. If we define $\delta = a - 1$, α_κ can be approximated by

$$\alpha_\kappa \simeq \delta^{1-\kappa/N} \qquad \text{and} \qquad \prod_{\kappa=0}^{q-1} \alpha_k \simeq \delta^{q-(1/2n)q(q-1)} \tag{7.23}$$

Keeping only the leading nonvanishing order of Eq. (7.21) the amplitude of the spectral lines $v_0 = \pi$, $v_1 = \pi/2$, $v_q = \pi \cdot q \cdot 2^{-M}$ becomes

$$|A_N(2^\mu)| = \frac{1}{2}, \qquad |A_N(2^{\mu-1})| = \frac{1}{2}$$
$$|A_N(2^\mu q)| = \frac{1}{2}\delta^M \prod_{\kappa=0}^{M-1} |\sin \pi 2^{-M-1+\kappa} q| \tag{7.24}$$

($M = N - \mu$), respectively. q is odd and smaller than 2^M and M is sufficiently smaller than N. Approximating in a similar way the amplitude of spectral lines newly brought into existence (i.e., $v_q = \pi \cdot q^{2-N}$), we obtain

$$|A_M(q)| \simeq \tfrac{1}{2}(1 + \eta[a^{2^N} - 1])\, a^{2^{N-1}} \delta^{N/2} \prod_{\kappa=0}^{N-2} |\sin \pi q 2^{-N+\kappa}| \tag{7.25}$$

Already after a few iterations, expression (7.25) approaches expression (7.24). The constant η varies between $-1/2$ and $(1 - 2\sqrt{2})/4$ when the control parameter a is decreased from $2^{2^{-N+1}}$ to $2^{2^{-N}}$. At each band splitting η has to be reset to $\eta = -1/2$. Inspecting equation (7.21), one finds that the power spectrum is continuous with respect to the control parameter: At a band-splitting point, the last term of equation (7.21) can be split into two parts: one of them increases the upper index of the second sum by 1, whereas the other one contains the reduced η.

The power spectrum looks rather similar to the power spectrum of the logistic map with the exception that the intensity of each line is scaled by a factor δ^M, which vanishes in the limit $a \to 1$. If we decrease $a \to 1$, the bands become extremely small and concentrated to a tiny neighborhood of the points 0 and 1. This is the reason why only the spectral lines $v = \pi$ and $v = \pi/2$ are independent of δ. If we transform the dynamical system [Eq. (2.1)] back to the original tent map [Eq. (2.11)], all spectral lines but $v = \pi$ are multiplied by an additional factor δ.

8. CONCLUSIONS

In the preceding sections, we have shown that the Frobenius–Perron operator H of even a quasilinear map has an amazingly complex spectrum. The qualitative results of this paper can be transferred to Frobenius Perron operators induced by a wide class of endomorphisms (a typical example is the logistic map). The crucial point is that the operators, although bounded, are neither normal nor compact. The preimages of the critical point are constructive elements of an infinite sequence of null spaces. Linear combinations of elements of these spaces can be used to build up eigenfunctions of H to any $\lambda \in C$ restricted by $|\lambda| < 1$. Usually, functions in the null spaces have a support smaller than the unit interval and the supports of different null spaces have nonempty intersections. The eigenfunctions constructed in this way will be discontinuous in general. If, however, the invariant density is piecewise constant with a finite number of jumps, then there are continuous and even differentiable eigenfunctions.

Maps with a piecewise constant invariant density are equivalent to finite Markov chains. The intervals of constant density define a natural partition of the unit interval and the Frobenius–Perron operator becomes matrix valued. It is easy to determine a right and left set of eigenvectors of the matrix part and to find a complete base of eigenfunctions in each interval. Actually, it suffices to determine a base of eigenfunctions of that interval which contains the critical point, since the bases of all other intervals can be obtained by linear transformations. The invariant density of the critical interval is constant, so it must be conjugate to the map with control parameter $a = 2$. The polynomial eigenfunctions of this map are Bernoulli's polynomials $B_{2n}(x/2)$ (nonzero eigenvalues) and $B_{2n+1}(x)$ (null space). It is clear that these eigenfunctions are the most appropriate ones for an expansion of polynomial correlation functions like $\langle x(n)\, x \rangle$ or $\langle x^2(n)\, x^2 \rangle$. Each interval has its own base and the calculation of the various coefficients of the expansion is a simple exercise in linear algebra.

This scheme fails if the invariant density has an infinite but countable number of steps. The spectrum of the Frobenius–Perron operator consists now of four parts: (1) $\lambda = 0$ is a point of accumulation, corresponding to the null space N_0; (2) there is a countable number of eigenvalues in the interior of the annulus $1/a < |\lambda| < 1$; (3) the circle $|\lambda| = 1/a$ belongs to the continuous spectrum, since there is a sequence of approximate eigenfunctions $\{P_n\}$, $\|P_n\| = 1$ such that $\lim \|HP_n - \lambda P_n\| \to 0$; (4) the entire interior of the disk $|\lambda| = 1$ is part of the essential spectrum of H corresponding to eigenfunctions which consist of an infinite sum of similar elements of the null spaces N_i. (Note, that Bernoulli's polynomials of the periodic case $a = 2$ belong to this category.) We were able to show that the second part of the spectrum is not sensible to a small change of the control parameter. The isometric structure of the adjoint operator H^+ allows only for eigenfunctions with eigenvalues on the unit circle of the complex plane. Thus the usual well-known procedure of expanding an arbitrary initial distribution from $L_2(\mu)$ in terms of eigenfunctions of H fails unless we restrict ourselves to the nondecaying part of the asymptotic behavior. We are, on the other hand, not really interested in the left eigenfunctions of H but only in a projection formalism which allows for the expansion of an arbitrary probability density in terms of eigenfunctions of H. It was shown in Section 6, that such a formalism exists and that the calculation of the expansion coefficients can be reduced to a Riemann–Stieltjes integral over a continuous function $\Omega(x, z_l)$ with the initial distribution occurring in the differential. If the eigenvalue λ_l has modulus 1, an integration by part transforms the integral in the well-known inner product of a left eigenfunction and an arbitrary initial distribution. This formalism is restricted to function of bounded variation because only for those functions the existence of the integrals is guaranteed. It also shows that any function of bounded variation decays exponentially to the space of piecewise constant functions with jumps in forward direction. Near the band-splitting points, the decay is extremely slow and for parameter values $a < \sqrt{2}$ the map lost its mixing property. The slow relevant eigenvalues can be seen as a first indication of band splittings which will take place at smaller parameter values of a. The short-time behavior, on the other hand, is determined by the Lyapunov exponent as the characteristic inverse time constant.

Besides these well-behaved functions, there is an uncountable set of pathological functions. It will be shown in a subsequent paper[11] that all eigenfunctions with eigenvalue $|\lambda_1| > 1/a$ constructed from an infinite sequence of null space based functions are of unbounded variation and it is even possible to show that their capacity dimension lies between 1 and 2. There are even more pathological functions of capacity dimension 2, which decay algebraically.

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